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Symmetry of anomalous dimension matrices explained

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Abstract: In a previous paper, one of us pointed out that the anomalous dimension matrices for all physical processes that have been calculated to date are complex symmetric, if stated in an orthonormal basis. In this paper we prove this fact and show that it is only true in a subset of all possible orthonormal bases, but that this subset is the natural one to use for physical calculations.

Keywords: Jets, QCD.

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1. Introduction

In perturbation theory resummation becomes necessary when large logarithms compensate the smallness of the coupling constant and invalidate a fixed order calculation.

For coloured processes the issue of resummation is complicated by the non-Abelian colour structure. For soft wide angle radiation, where real and virtual gluon emission cancel (at least for global observables), this complication can be expressed in terms of a matrix containing colour and phase space information, the soft anomalous dimension matrix [1−11],

$$\mathbf{\Gamma} = \sum_{i \neq j} \mathbf{T}_i \cdot \mathbf{T}_j \; \Omega_{ij}, \tag{1.1}$$

where $\mathbf{T}_i \cdot \mathbf{T}_j$ denotes the effect of exchanging a gluon between partons i and j on the colour structure and Ω_{ij} is the result of integrating (azimuth and rapidity) over the region of phase space where emission is forbidden. This integral can have an imaginary part, coming from Coulomb phase contributions.

It has previously been observed that the soft anomalous dimension matrices have always been symmetric if stated in orthonormal bases [12]. Here we give the proof. The outline of the proof will be to first show that in orthonormal bases the colour matrix $\mathbf{T}_i \cdot \mathbf{T}_j$ is Hermitian. Then we show that for a particularly natural choice of basis, it is also real. From this the symmetry of $\mathbf{T}_i \cdot \mathbf{T}_j$, and hence $\mathbf{\Gamma}$, follows. We start by setting the notation.

2. Notation

We consider the amplitude, $\mathcal{M}$, for a hard scattering process involving m partons (we do not need to make the distinction between incoming and outgoing partons for the colour considerations discussed here). In general it is a function of the colour index of each of the m partons,

$$\mathcal{M} = \mathcal{M}_{a_1 \ldots a_i \ldots a_j \ldots a_m}, \tag{2.1}$$

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where a_i is in the range 1 to N_c if i is a quark or antiquark and 1 to N_c^2-1 if i is a gluon. In the present discussion, it is only the colour of partons i and j that will be relevant, so we suppress all other colour labels. The set of physical colour states forms a vector space, and it is useful to introduce a bra-ket notation for this space,

$$|M\rangle \equiv \mathcal{M}_{\dots a_i \dots a_j \dots}. \tag{2.2}$$

Physical cross sections are given by the interference of amplitudes, summed over colour indices,

$$\sigma \sim \sum_{\dots a_i \dots a_j \dots} \mathcal{N}^*_{\dots a_i \dots a_j \dots} \mathcal{M}_{\dots a_i \dots a_j \dots} \equiv \langle N|M\rangle. \tag{2.3}$$

We are interested in the effect on the colour state of $\mathcal{M}$ of exchanging a gluon between partons i and j, which can be written

$$T^\alpha_{a_i b_i} T^\alpha_{a_j b_j} \mathcal{M}_{\dots b_i \dots b_j \dots}, \tag{2.4}$$

where α is the colour index of the exchanged gluon and T is the generator of the fundamental representation, $t^\alpha_{a_i b_i}$, if i is an outgoing quark or incoming antiquark, its complex conjugate, $(t^\alpha_{a_i b_i})^* = t^\alpha_{b_i a_i}$, if i is an outgoing antiquark or incoming quark, and the generator of the adjoint representation, $if_{a_i \alpha b_i}$, if i is a gluon.[1] Using the Hermiticity of the generators, the effect of gluon exchange on the conjugate amplitude is

$$(T^\alpha_{a_i b_i})^* (T^\alpha_{a_j b_j})^* \mathcal{M}^*_{\dots b_i \dots b_j \dots} = \mathcal{M}^*_{\dots b_i \dots b_j \dots} T^\alpha_{b_i a_i} T^\alpha_{b_j a_j}. \tag{2.5}$$

Similarly, the interference of an amplitude formed by emitting a gluon from parton i in an amplitude $\mathcal{N}$ with that formed by emitting a gluon from parton j in an amplitude $\mathcal{M}$ is given by

$$\mathcal{N}^*_{\dots b_i \dots b_j \dots} T^\alpha_{b_i a_i} T^\alpha_{b_j a_j} \mathcal{M}_{\dots a_i \dots a_j \dots}. \tag{2.6}$$

It is therefore natural to define a colour operator $\mathbf{T}_i \cdot \mathbf{T}_j$, to represent the exchange of a gluon between partons i and j, independent of where it lies relative to the cut through the diagram that defines the physical final state,

$$\mathcal{N}^*_{\dots b_i \dots b_j \dots} T^\alpha_{b_i a_i} T^\alpha_{b_j a_j} \mathcal{M}_{\dots a_i \dots a_j \dots} \equiv \langle N|\mathbf{T}_i \cdot \mathbf{T}_j|M\rangle = \langle N|\mathbf{T}_j \cdot \mathbf{T}_i|M\rangle. \tag{2.7}$$

The symmetry of the colour operator in its indices i and j, $\mathbf{T}_i \cdot \mathbf{T}_j = \mathbf{T}_j \cdot \mathbf{T}_i$ is obvious from its definition (2.7) – since the two generators act on different partons' indices their order is irrelevant.

In practical calculations it is convenient to introduce a basis for the vector space of possible colour states for the hard process, the elements of which we denote $|K\rangle$. Any state $|M\rangle$ can be written as a linear combination of $|K\rangle$ states,

$$|M\rangle \equiv |K\rangle \mathrm{M}_K, \tag{2.8}$$

[1]Different sign conventions are often chosen for physical calculations, but that will not be relevant for the present discussion.

where M_K are a set of complex numbers (represented as a column vector M). Restricting ourselves to orthonormal bases,[2] we have

$$\mathrm{M}_K = \langle K|M\rangle. \tag{2.9}$$

Our aim is to find, and categorize the properties of, the matrix representation of $\mathbf{T}_i \cdot \mathbf{T}_j$. Acting with $\mathbf{T}_i \cdot \mathbf{T}_j$ on the state $|M\rangle$ defines a new state, $\mathbf{T}_i \cdot \mathbf{T}_j|M\rangle$ from which we obtain

$$(\mathbf{T}_i \cdot \mathbf{T}_j)_{LK} = \langle L|\mathbf{T}_i \cdot \mathbf{T}_j|K\rangle. \tag{2.10}$$

We can now state the goal of our proof. Since we observe that the matrix representation of Γ is symmetric for arbitrary observables, and the Ω_{ij} are observable-dependent, it must be that the matrix representation of $\mathbf{T}_i \cdot \mathbf{T}_j$ is symmetric in orthonormal bases. Our proof is in two parts, first we prove that it is Hermitian in any orthonormal basis, then we try to prove that it is real, since a real Hermitian matrix is automatically symmetric. However, we will find that it is true in only a subset of bases. Finally, we will argue that these are the most natural set of bases, and hence that it is not surprising that all previous calculations, when orthonormalized, do give symmetric matrices.

3. Hermiticity of colour structure

Having taken care to set up the notation, the proof that the matrix representation of $\mathbf{T}_i \cdot \mathbf{T}_j$ is Hermitian is straightforward. It is a consequence only of the definition (2.10) and the Hermiticity of the generators in the parton indices. In fact it is already clear from the definition of the matrix element (2.7) that the right-hand-side of eq. (2.10) is Hermitian,

$$\begin{aligned}
((\mathbf{T}_i \cdot \mathbf{T}_j)_{LK})^* = (\langle L|\mathbf{T}_i \cdot \mathbf{T}_j|K\rangle)^* &= (\mathcal{L}^*_{\dots b_i \dots b_j \dots} T^\alpha_{b_i a_i} T^\alpha_{b_j a_j} \mathcal{K}_{\dots a_i \dots a_j \dots})^* \\
&= \mathcal{K}^*_{\dots a_i \dots a_j \dots} T^\alpha_{a_i b_i} T^\alpha_{a_j b_j} \mathcal{L}_{\dots b_i \dots b_j \dots} \\
&= \langle K|\mathbf{T}_i \cdot \mathbf{T}_j|L\rangle = (\mathbf{T}_i \cdot \mathbf{T}_j)_{KL}.
\end{aligned} \tag{3.1}$$

That is,

$$(\mathbf{T}_i \cdot \mathbf{T}_j)^\dagger = \mathbf{T}_i \cdot \mathbf{T}_j, \tag{3.2}$$

the matrix representation of $\mathbf{T}_i \cdot \mathbf{T}_j$ is Hermitian.

4. Realness of scalar product

It now remains only to show that $\langle K|\mathbf{T}_i \cdot \mathbf{T}_j|L\rangle$ is real for all i and j for all elements $|K\rangle$, $|L\rangle$ in an orthonormal basis. In fact it is easy to see that this cannot be the case in general, because if there is some initial basis in which it is true, one can apply an arbitrary unitary

[2]Anomalous dimension calculations have typically been performed in orthogonal, but not normalized, bases, in which the symmetry property observed in [12] is not manifest. It is interesting to note that [13], which appeared on the same day as [12], does use an orthonormal basis and does obtain a symmetric anomalous dimension matrix, as does an appendix, not referred to in the rest of the paper, of [14], which appeared some weeks later.

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transformation to another orthonormal basis in which it is not. Our task is therefore to show that there does exist a basis, or set of bases, in which it is true. In this section we show, not only that this is the case, but also that such a basis is the most natural one to use for a physical calculation, explaining why this property has been observed in all previous calculations.

In general, one can always choose a basis constructed from delta functions in quark indices, delta functions in gluon indices, generators t^a_{bc}, and the symmetric and antisymmetric structure constants d_{abc} and if_{abc},

$$(if/d)_{abc} = 2(\mathrm{Tr}[t^a t^b t^c](-/+)\mathrm{Tr}[t^b t^a t^c]). \tag{4.1}$$

These can be used to form a complete basis for any physical process, because the Feynman rules contain no other factors. Therefore the matrix element $\langle K|\mathbf{T}_i \cdot \mathbf{T}_j|L\rangle$ is a scalar in colour space, constructed from these building blocks. Using the relation (4.1), all occurences of d_{abc} and if_{abc} can be converted into generators of the fundamental representation. Moreover, these generators can be replaced pairwise by delta functions using the relation

$$t^\alpha_{ca} t^\alpha_{db} = \frac{1}{2}\left(\delta_{ad}\delta_{bc} - \frac{1}{N_c}\delta_{ac}\delta_{bd}\right). \tag{4.2}$$

Thus every matrix element in such a basis reduces to strings of delta functions with real coefficients and is clearly real.

Therefore any orthonormal basis constructed from delta functions, generators and group structure constants will result in a symmetric anomalous dimension matrix. As we have already mentioned, such a choice of basis is extremely natural for describing physical processes, since the colour part of any Feynman rule can be represented in this way. Although we have shown that the colour part of the anomalous dimension matrix, $\mathbf{T}_i \cdot \mathbf{T}_j$ in eq. (1.1), is real and Hermitian, and therefore symmetric, the kinematic part, Ω_{ij}, is complex in general, so the anomalous dimension matrix itself is complex symmetric, and hence not Hermitian.

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